Int. J. Contemp. Math. Sciences, Vol. 4, 2009, no. 24, 1149 - 1157

A Study on a Subset of Absolutely Convergent Sequence Space

U. K. Misra[1], M. Misra[2], N. Subramanian[3] and P. Samanta[4]

[1]Department of Mathematics, BerhampurUniversity
Berhampur-760 007, Orissa, India
umakanta_misra@yahoo.com

[2]Principal, Government Science College
Malkangiri, Orissa, India

[3]Department of Mathematics, Sastra University
Tanjore-613 402, Tamilnadu, India

[4]Department of Mathematics, Gopalpur College
Gopalpur on sea, Orissa,India

Abstract: In this paper, we define the section sequence space ℓ_s which is called the section sequence space of ℓ and study the inclusion $\ell_s \subset \ell$. Further AK-property, Dual space of ℓ_s are studied.

Mathematics Subject Classification: 40A05

Keywords: Schauder basis, dual space, solid, weak and strong convergence, separable

1. Introduction and preliminaries

Let ℓ be an BK-space. We denote ℓ_s as the sequence consisting of all those sequences $\ell_s = \{x = (x_k) : (y_k) \in \ell\}$, where

$y_k = x_1 + x_2 + x_3 + . . . + x_k$. , for each fixed $k = 1,2,3,\ldots$.

For a sequence $(y_k) \in \ell_s$, we can calculate the sequence (x_k) by

$x_1 = y_1$,

$x_2 = y_2 - x_1 = y_2 - y_1.$, $x_3 = y_3 - x_1 - x_2 = y_3 - y_1 - (y_2 - y_1) = y_3 - y_2 \ldots.$

$x_n = y_n - y_{n-1}$.

For any $x \in \ell_s$, we define

$$\|x\|_s = \{|x_1| + |x_1 + x_2| + \ldots + |x_1 + x_2 + \ldots + x_k + x_{k+1}| + \ldots\} < \infty \quad .$$

For a given a sequence $x = \{x_k\}$, we define the n^{th} section as the sequence

$$x^{(n)} = \{x_1, x_2, . . . , x_n, 0,0,. . . \} .$$

Let

$$\delta^{(n)} = (0,0,\ldots,1,-1,0,0,0,\ldots),$$

where 1 is in the n^{th} place and -1 in the $(n+1)^{th}$ place .

An FK-space X is said to have AK-property if $\{\delta^{(n)}\}$ is a Schauder basis for X .The space X is said to have AD if Φ is dense in X . We note that AK$\Rightarrow$AD by [1] .

For the sequence space X , we define

$$X^{\beta} = \left\{a = \{a_k\} : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for each } x \in \text{X}\right\}$$

We called X^{α}, X^{β}, X^{γ} as the α-dual of X , β-dual of X , γ-dual of X, respectively. Note that $X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$. If $X \subset Y$ then $Y^{\mu} \subset X^{\mu}$, for $\mu = \alpha , \beta$ *and* γ .

We have the following known results.

Lemma 1: (See Theorem 7.2.7.in [3]

Let X be an FK-space $\supset \Phi$. Then

(i) $\text{X}^{\gamma} \subset X^{f}$

(ii) If X has AK, $X^{\beta} = X^{f}$

(iii) If X has AD, $X^{\beta} = X^{\gamma}$

Lemma 2 (Page 69, 2.3.1 in [2]):

If a Normed space X has a Schauder basis, then X is separable.

2. Main Results:

In this section we study some of the property of ℓ_s.

***Proposition*-1**: ℓ_S has Schauder basis namely $(e_1, e_2, e_3, \ldots e_k, \ldots)$, where $e_k = \{0,0,0,\ldots,1,-1,0,0,\ldots\}$, 1 is in the k^{th} place and -1 is at the $(k+1)^{th}$ for k=1,2,….

Proof. We know that $\{\delta^{(1)}, \delta^{(2)}, \ldots\}$ is a Schauder basis for ℓ transformations given in the introduction. It follows that $(e_1, e_2, e_3, ..., e_k)$ is a Schauder basis for ℓ_S.

Theorem- 1 ; ℓ_S has AK-property.

Proof. Let $x = (x_k) \in \ell_s$. Then $\mathrm{T}(y_k) \in \ell$ with $y_k = x_1 + x_2 + \ldots + x_k$.

Put $x^{(n)} = (x_1, x_2, x_3, \ldots, x_n, 0, 0, \ldots)$. Then

$$\left\|x - x^{(n)}\right\| = \|0,0,0,\ldots, x_{n+1}, x_{n+2}, \ldots\| = |x_{n+1}| + |x_{n+1} + x_{n+2}| + \ldots$$

$$= |y_{n+1} - y_n| + |y_{n+2} - y_n| + |y_{n+3} - y_n| + \ldots$$

$$= \sum_{k=n+1}^{\infty} |y_k - y_n| \to 0 \text{ , as } n \to \infty.$$

$$= \sum_{k=n+1}^{\infty} |y_k| \to 0 \text{ , as } n \to \infty, \text{ becuase } (y_k) \in \ell.$$

Thus we have $0 \le \left\|x - x^{(n)}\right\| \le 0$, for sufficiently large *n*.. Hence $\left\|x - x^{(n)}\right\| \to 0$, as $n \to \infty$,. Therefore the space ℓ_S has AK. This completes the proof.

Corollary -1: The set $\{\delta^{(1)}, \delta^{(2)}, \ldots\}$ is a Schauder basis for ℓ_S.

Proof:. By p.59,4.2.13 in [3].

Proposition- 2: $\ell_s \subset \ell$ and the inclusion is strict.

Proof. Let $x_k \in \ell_s$. Then $y_k \in \ell$. Hence $\sum_{k=1}^{\infty} |y_k| < \infty$. But as $x_k = y_k - y_{k-1}$. We have

$$|x_k| = |y_k - y_{k-1}| \leq |y_k| + |y_{k-1}|$$

Then

$$\sum_{k=1}^{\infty} |x_k| \leq \sum_{k=1}^{\infty} |y_k| + \sum_{k=1}^{\infty} |y_{k-1}|$$

Hence $x_k \in \ell$. Consequently $\ell_s \subset \ell$.

Next we show that he above inclusion is strict. For this take the sequence $\delta^{(1)} = (1,0,0,\ldots)$. Then $\delta^{(1)} \in \ell$ and thus we have

$y_1 = 1$, $y_2 = 1+0 = 1$, $y_3 = 1+0+0 = 1, \ldots, y_k = 1+0+\ldots+0 = 1$. Now, $|y_k| = 1$ for all k. Hence $(|y_k|)$ does not tend to zero as $k \to \infty$. Hence $\delta^{(1)} \notin \ell_s$. Thus the inclusion $\ell_s \subset \ell$. This completes the proof.

heorem-2 : The dual of space ℓ_s is ℓ_∞.

Proof: A Schauder basis for ℓ_s is (e_k) where $e_k = (s^k)$ has 1 in the k^{th} place and -1 in the $(k+1)^{\text{th}}$ place and zero's elsewhere. Let $x \in \ell_{ss}$. Then there exist scalars $\alpha_1, \alpha_2, \ldots..$ such that $x = \sum_{k=1}^{\infty} \alpha_k e_k$ is unique. Now for any bounded linear operator f on ℓ_s we have

$$f(x) = f(\sum_{k=1}^{\infty} \alpha_k e_k) = \sum_{k=1}^{\infty} \alpha_k f(e_k) = \sum_{k=1}^{\infty} \alpha_k \gamma_k ,$$

where the numbers $\gamma_k = f(e_k)$ are uniquely determined by *f*. Also $\gamma_k = f(e_k), |\gamma_k| = |f(e_k)|$. Since f is linear and bounded $|\gamma_k| = |f(e_k)| \leq \|f\|_s \|e_k\|_s$. But

$$\|e_k\|_s = \|s^{(k)}\|_s = \|(0,0,\ldots,1,-1,0,0,\ldots)\| = |0| + |0+0| + \ldots + |0+1| + |1-1| + \ldots .$$

(sum of the first *k* terms) and $\|s^{(k)}\|_s = \|e_k\|_s = |1| = 1$. Thus

$$|\gamma_k| \le \|f\|_s \|e_k\|_s \le \|f\|_s .1$$

$$\Rightarrow |\gamma_k| \le \|f\|_s \quad \Rightarrow \sup_{(k)} |\gamma_k| \le \|f\|_s = M.$$

Hence $(\gamma_k) \in \ell_\infty$. Therefore

(2.1) $$\ell_s^{'} \subset \ell_\infty.$$

But by Proposition- 2, $\ell_s \subset \ell$. Hence $\ell^{'} \subset \ell_s^{'}$. As $\ell^{'} = \ell_\infty$,

(2.2) $$\ell_\infty \subset \ell_s^{'}$$

Hence from (2.1) and (2.2) $\ell_s^{'} = \ell_\infty$.This completes the proof.

Theorem-3 : The β-dual of ℓ_s is ℓ_∞ .

Proof: By Proposition-2 we get $\ell_s \subset \ell$. Hence $\ell^\beta \subset (\ell_s)^\beta$. But $\ell^\beta = \ell_\infty$. Hence

(2.3) $$\ell_\infty \subset (\ell_s)^\beta.$$

Next, let $y \in (\ell_s)^\beta$ and $f(x) = \sum_{k=1}^{\infty} x_k y_k$ with $x \in \ell_s$. Take $x = s^{(k)} \in \ell_s$, where $s^{(k)} = (0,0,\dots,1,-1,0,\dots)$, $\{|x_n|\} = \{0,0,0,\dots,1,0,\dots\}$. As this converges to zero, $s^{(k)} \in \ell_s$. Hence

$$\|s^{(k)}\| = \left\{ \begin{matrix} |0| + |0+0| + |0+0+0| + \dots + \\ |0+0+\dots+1| + |0+0+\dots+1-1| + \dots \end{matrix} \right\}$$

$$\Rightarrow \qquad \|s^{(k)}\| = 1 .$$

But

(2.4) $$|y_n| = \left| f\left(s^{(k)}\right) \right| \le \|f\| \|s^{(k)}\| \le \|f\|.1 = \|f\| .$$

Thus $\{y_n\}$ is a bounded sequence. Further,as y is arbitrary in $(\ell_s)^\beta$.

(2.5) $$(\ell_s)^\beta \subset \ell_\infty.$$

From (2.3) and (4.3) we get $(\ell_s)^\beta = \ell_\infty$. This completes the proof.

Proposition-3: ℓ_S is solid .

Proof: Let $|x_k| \leq |y_k|$ with $y = (y_k) \in \ell_s$. So $|\xi_k| \leq |\eta_k|$ with $\eta = (y_k) \in \ell$. But ℓ is solid. Hence $\xi = (\xi_k) \in \ell$ .Therefore $x = (x_k) \in \ell$ .Hence ℓ_s is solid. This completes the proof.

Corollary-2: In ℓ_s , weak convergence does not imply strong convergence.

Proof: Assume that weak convergence implies strong convergence in ℓ_s . Then we would have $(\ell_s)^{\beta\beta} = \ell_s$ [see(1)]. But $(\ell_s)^{\beta\beta} = (\ell_\infty)^{\beta} = \ell$. By Proposition 2. ℓ_s is a proper subspace of ℓ. Thus $(\ell_s)^{\beta\beta} \neq \ell_s$. Hence weak convergence does not imply strong convergence in ℓ_s .

This completes the proof.

Corollary-3: $(\ell_S)^{\mu} = \ell_\infty$ where $\mu = \alpha, \beta, \gamma, f$.

Proof: ℓ_s has AK property, by theorem- 1. Hence by Theorem- 7.3.9 in [3] we get $(\ell_s)^{\beta} = (\ell_s)^{f}$. But $(\ell_s)^{\beta} = \ell_\infty$ .Hence

(2.6) $$(\ell_s)^{f} = \ell_\infty .$$

Since AK $\Rightarrow$ AD , from [3] we get $(\ell_s)^{\beta} = (\ell_s)^{\gamma}$. Therefore

(2.7) $$(\ell_s)^{\gamma} = \ell_\infty.$$

By proposition-3, we have ℓ_s is solid. Hence by Theorem 7.3.9 in [3], We get

(2.8) $$(\ell_s)^{\alpha} = (\ell_s)^{\gamma} = \ell_\infty .$$

From (2.6), (2.7) and (2.8) ,we have $(\ell_s)^{\alpha} = (\ell_s)^{\beta} = (\ell_s)^{\gamma} = (\ell_s)^{f} = \ell_\infty$.

This completes the proof.

Theorem -4: Let Y be any FK-space $\supset \Phi$. Then $Y \supset \ell_s$ if and only if $\{\delta^{(k)}\}$ is weakly bounded.

Proof: In order to establish the result it is enough to establish the following result:

$$Y \supset \ell_s \Leftrightarrow Y^{f} \subset (\ell_s)^{f} .$$

Since ℓ_s has AD and $(\ell_s)^{f} = \ell_\infty$, by using Theorem 8.6.1 in [3] we have

$$Y^{f} \subset \ell_\infty .$$

$\Leftrightarrow$ for each $f \in Y^{'}$, the topological dual of Y $\Leftrightarrow$ $f\left(\delta^{(k)}\right) \in \ell_{\infty}$

$\Leftrightarrow$ $f\left(\delta^{(k)}\right)$ is bounded $\Leftrightarrow$ *Thesequence* $\{\delta^{(k)}\}$ *is weakly bounded* .

This completes the proof.

Theorem-5: In ℓ_s ,weakly convergent sequences are norm convergent.

Proof: Let $a = \{a_1, a_1 + a_2, a_1 + a_2 + a_3, ...\}$ be weakly convergent and let $A = (a_{nk})$ be an infinite matrix. Let us assume that A is coercive. Since $(\ell_s)^{'} = \ell_{\infty}$, it is a conservative matrix.So the column exists by, Theorem 1.3.6 in [3]. By using Theorem 1.3.7 in [3]

$$|a_1, a_1 + a_2, a_1 + a_2 + a_3, \ . \ . \ . \ | = \lim_{n \to \infty} \begin{Bmatrix} |a_{n1}| + |a_{n1} + a_{n2}| + \\ |a_{n1} + a_{n2} + a_{n3}| + . \ . \ . \end{Bmatrix}$$

$$\leq \|A\|.$$

Since bounded monotonic sequence converges, $a \in \ell_s$.

This completes the proof.

Proposition-4: ℓ_s is not perfect.

Proof: We know that $(\ell_s)^{'} = \ell_{\infty}$ .Hence $(\ell_s)^{''} = (\ell_{\infty})^{'}$ .But as $(\ell_{\infty})^{'} = \ell$, $(\ell_s)^{''} = \ell$. Hence ℓ_s is not perfect. This completes the proof.

Proposition-5: The space ℓ_s is separable.

Proof: By Proposition 1,we have ℓ_s has Schauder basis $\{e_1, e_2, ..., e_n, ...\}$.. Also ℓ_s is a Banach space. Hence, by the Lemma- 2 , it follows that ℓ_s is separable . This completes the proof.

Proposition-6: The space ℓ_{∞} is not separable.

Proof: By Theorem 1.3.9 in [2].

Proposition-7: The space ℓ_s is not reflexive.

Proof. By Proposition-5, we have ℓ_s is separable. But, by Proposition 2 $(\ell_s)^{'} = \ell_{\infty}$. Since ℓ_{∞} is not separable by Proposition-6, ℓ_s is not reflexive.This completes the proof.

Theorem-6: The space ℓ_s is an inner product space but not a Hilbert space.

Proof. The proof will be established by showing that the norm satisfies the law of parallelogram . Let us take

$$x = \{1,-1,0,\ldots\} \in \ell_s \text{ and } y = \{1,-1,0,\ldots\} \in \ell_s .$$

Then

$$\|x\|_s = \{|x_1| + |x_1 + x_2| + |x_1 + x_2 + x_3| + \ldots\}$$

$$= \{|1| + |1-1| + |1-1+0| + \ldots\} = \{1+0+0+\ldots\} = 1$$

Similarly,

$$\|y\|_s = \{|1| + |1-1| + |1-1+0| + \ldots\} = \{1+0+0+\ldots\} = 1$$

Consider,

$$\|x+y\|_s = \{|x_1 + y_1| + |(x_1+y_1) + (x_2+y_2)| + \ldots\} = 2$$

Similarly,

$$\|x-y\|_s = \{|x_1 - y_1| + |(x_1-y_1) + (x_2-y_2)| + |(x_1-y_1) + (x_2-y_2) + (x_3-y_3)| + \ldots\}$$
$$= 0.$$

Now

$$\Rightarrow 2^2 + 0 = 2\{1^2 + 1^2\} \quad \Rightarrow 4 = 4 \quad .$$

Thus parallelogram law is satisfied. Therefore ℓ_s is an inner product space.

For the proof of the second part let us suppose that ℓ_s is a Hilbert space. Then by [2] (Theorem 4.6.6) ℓ_s would satisfy reflexivity condition . This contradicts Proposition-7. Hence ℓ_s is not a Hilbert space. This completes the proof.

REFERENCES

[1] Brown, H. I, The summability field of a perfect $\ell - \ell$ method of summation Journal D'Analyse Mathematique, 20 (1967), 281-287.

[2] Erwin, Kreyszig, Introductory Functional Analysis with Applications, John Wiley & Sons Inc, 1978.

[3] Wilansky, A, Summability through Functional Analysis, North Holland Mathematics Studies, Vol.85, North-Holland ,Amsterdam, 1984.

Received: November, 2008